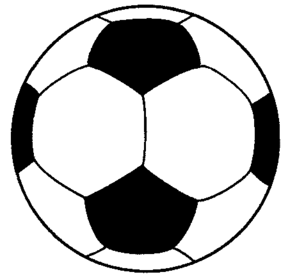
Chapter 6.

Matrices

Situation.

A league soccer competition involves six teams:

Ajax, Battlers, Cloggers,
Devils, Enzymes, Flames.



Each team plays one game per week and, during the ten week competition, plays each other team twice, once in the first five weeks and once in the last five weeks. (All teams play at the same venue so no consideration needs to be made to balance home and away games.)

The results for the first five weeks gave rise to the following table:

	Played	Won	Drawn	Lost	Goals scored	
					For	Against
Ajax	5	2	1	2	10	5
Battlers	5	2	1	2	4	5
Cloggers	5	2	0	3	7	6
Devils	5	2	0	3	4	11
Enzymes	5	3	2	0	8	2
Flames	5	2	0	3	5	9

✍ Create a similar table for the last five weeks of the competition using the results stated below.

Week 6
Ajax 3 1 Battlers
Cloggers 4 1 Devils
Enzymes 5 4 Flames

Week 7
Ajax 1 2 Cloggers
Battlers 1 0 Enzymes
Devils 1 1 Flames

Week 8
Ajax 2 2 Devils
Battlers 1 0 Flames
Cloggers 4 3 Enzymes

Week 9
Ajax 0 1 Enzymes
Battlers 2 0 Devils
Cloggers 1 0 Flames

Week 10
Ajax 1 3 Flames
Battlers 0 1 Cloggers
Devils 0 1 Enzymes

✍ Create a table like the one above for the complete ten week competition.

As part of the soccer league activity on the previous page we had to arrange information in a "rows and columns" form of presentation.

This rows and columns *rectangular array* presentation of numbers is called a **matrix**. (Plural: matrices).

If we remove the headings and indicate the start and end of the matrix with brackets, the table given on the previous page would be written as shown on the right:

$$\begin{bmatrix} 5 & 2 & 1 & 2 & 10 & 5 \\ 5 & 2 & 1 & 2 & 4 & 5 \\ 5 & 2 & 0 & 3 & 7 & 6 \\ 5 & 2 & 0 & 3 & 4 & 11 \\ 5 & 3 & 2 & 0 & 8 & 2 \\ 5 & 2 & 0 & 3 & 5 & 9 \end{bmatrix}$$

This matrix has 6 rows and 6 columns. We say it is a *six by six* matrix, (written 6×6). This gives the **size** or **dimensions** of the matrix.

Matrices do not have to have the same number of rows as they have columns, however those that do are called **square matrices.**
The matrix on the right is a 6×6 square matrix.

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0{\cdot}5 \end{bmatrix}$$

2 rows and 3 columns.

A 2×3 matrix.

$$\begin{bmatrix} 2 & 5 \\ 11 & -2 \end{bmatrix}$$

2 rows and 2 columns.

A 2×2 matrix. (A square matrix.)

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

3 rows and 1 column.

A 3×1 matrix.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

3 rows and 4 columns.

A 3×4 matrix.

A matrix consisting of just one column, like the third matrix above, is called a **column matrix.**

Any matrix consisting of just one row is called a **row matrix**. For example:

$$\begin{bmatrix} 5 & 0 & -2 & 1 \end{bmatrix}$$

A square matrix having zeros in all spaces that are not on the **leading diagonal** is called a **diagonal matrix**.

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

We commonly use capital letters to label different matrices. The corresponding lower case letters, with subscripted numbers, are then used to indicate the row and column a particular entry or **element** occupies.

For the matrix A shown on the right the element occupying the 3rd row and 2nd column is the number 7.

$$A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 6 & -3 & 9 \\ 5 & 7 & 8 & 4 \end{bmatrix}$$

Thus $a_{32} = 7$.

Similarly $a_{11} = 2$,

$a_{12} = 0$,

$a_{13} = -1$, etc.

Adding and subtracting matrices.

In the soccer competition activity earlier in this chapter you probably determined the matrix for the full ten weeks by adding the matrix for the first five weeks to the matrix for the last five weeks. To perform such addition it was natural to simply add elements occurring in corresponding locations. This is indeed how we add matrices. For example,

$$\text{if } A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 3 & 5 \\ 2 & 1 & -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & -3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 2 & 0 & -5 \end{bmatrix} \text{ then } A + B = \begin{bmatrix} 3 & 1 & -1 & 5 \\ 9 & -1 & 5 & 9 \\ 5 & 3 & -3 & -1 \end{bmatrix}$$

$$\text{and similarly} \quad A - B = \begin{bmatrix} -1 & -1 & 5 & 1 \\ -1 & -3 & 1 & 1 \\ -1 & -1 & -3 & 9 \end{bmatrix}$$

Note: When adding or subtracting matrices there must be elements in corresponding spaces. Thus we can only add or subtract matrices that are the same size as each other.

Multiplying a matrix by a number.

Suppose that the 3 × 2 matrix shown on the right shows the cost of three models of gas heater in two different shops.

	Shop One	Shop Two
Economy	\$250	\$280
Standard	\$340	\$330
Deluxe	\$450	\$450

Now suppose that in a sale both shops offer 10% discount on all models.

The sale prices could be represented in a matrix formed by multiplying each element of the first matrix by 0·9.

	Shop One	Shop Two
Economy	\$225	\$252
Standard	\$306	\$297
Deluxe	\$405	\$405

This is indeed how we multiply a matrix by a number: We multiply each element of the matrix by that number. (This is referred to as "multiplication by a **scalar**".)

Equal matrices.

For two matrices to be equal they must be of the same size and have all corresponding elements equal.

Thus if $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 0 & -2 \end{bmatrix}$ then $a = 2$, $b = 3$, $c = -5$, $d = 1$, $e = 0$, $f = -2$

Example 1

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly and give the reason.

(a) A + B (b) A + C (c) B − A (d) 5C (e) 3B − 2A

(a)
$$A + B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

(b) A and C are not the same size. Thus A + C cannot be determined.

(c)
$$B - A = \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -6 \\ 1 & 4 & -7 \end{bmatrix}$$

(d)
$$5C = \begin{bmatrix} 10 & 15 \\ 5 & -25 \end{bmatrix}$$

(e)
$$3B - 2A = \begin{bmatrix} 9 & 15 & -6 \\ 3 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 8 \\ 0 & -8 & 10 \end{bmatrix} = \begin{bmatrix} 7 & 11 & -14 \\ 3 & 8 & -16 \end{bmatrix}$$

Many calculators will accept data in matrix form and can then manipulate these matrices in various ways.

Get to know the matrix capability of your calculator.

How does your calculator respond when you ask it to add together two matrices that are not of the same size?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & -2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

Exercise 6A

1. A matrix with m rows and n columns has size m × n. Write down the size of each of the following matrices.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \\ 1 & 0 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & -7 & 32 \\ 1 & 1 & 0 & 5 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 11 & -2 \end{bmatrix} \quad G = \begin{bmatrix} 12 & 3 \\ 0 & 5 \\ -5 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. If e_{mn} is the element situated in the m^{th} row and n^{th} column of matrix E determine
(a) e_{12} (b) e_{21} (c) f_{13} (d) g_{21} (e) g_{22} (f) g_{32}
where matrices E, F and G are as given below.

$$E = \begin{bmatrix} 5 & 4 & 13 \\ -4 & 2 & 0 \\ 1 & -8 & 12 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 5 & 7 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 \\ 7 & 3 \\ -2 & 0 \\ 4 & 11 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly.

(a) A + B (b) A + C (c) C − A (d) 2D
(e) 3B (f) B + D (g) 2A (h) 2A − C

4. If $P = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly.

(a) P + Q (b) Q − P (c) 3R (d) 3P − 2Q

5. If $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ determine each of the following. (If any cannot be determined state this clearly.)

(a) A + B (b) 3A (c) B + 2C (d) C + D

6. If $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & -3 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 1 & 3 & -1 \\ 2 & 1 & 4 & 3 \\ 1 & 5 & 2 & 0 \end{bmatrix}$ determine each of the following. (If any cannot be determined state this clearly.)

   (a) A + B (b) A + C (c) 2B (d) 5A − C

7. $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 7 \end{bmatrix}$.

   For each of the following write "Yes" if it can be determined and "No" if it cannot be determined.

   (a) A + B (b) B − A (c) 3C (d) A + D

   (e) A − 3D (f) A + 3B (g) B + B (h) A + B + C

8. Is matrix addition commutative? i.e. Does A + B = B + A (assuming A and B are of the same size).

9. Is matrix addition associative? i.e. Does A + (B + C) = (A + B) + C (assuming A, B and C are all of the same size).

10. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -7 & 12 \\ 1 & 0 & 13 \end{bmatrix}$ determine matrix C given that the following equation is correct:

$$3A - 2C = B.$$

11. For the first four games in a basketball season the points (P), assists (A), and blocks (B), that five members of one team carried out were as shown below.

	Game 1			Game 2			Game 3			Game 4		
	P	A	B	P	A	B	P	A	B	P	A	B
Alan	8	5	1	12	3	1	11	8	2	9	4	0
Bob	7	2	4	6	8	2	15	2	5	9	3	3
Dave	14	3	1	15	3	5	7	5	2	11	8	1
Mark	17	3	1	6	4	0	12	2	1	4	12	1
Roger	8	8	2	5	2	6	14	4	5	12	5	3

   (a) Construct a single 5 × 3 matrix showing the total points, total assists and total blocks each of these five players achieved for the 4 game period.

   (b) Construct a single 5 × 3 matrix showing the average points per game, average assists per game and average blocks per game for each of these five players for the 4 game period.

12. A company manufactures five types of lawn fertiliser:
   • Basic (B) • Feedit (F) • Fertilawn (FL) • Greenit (G) • Growgrass (GG)
   It sells these through its four garden centres.
   The number of bags of these fertilisers sold in these centres during the first and second halves of a year are given below:

   1st January → 31st June

	B	F	FL	G	GG
Centre I	3100	550	1040	820	2250
Centre II	1640	420	720	480	1480
Centre III	2850	520	1320	640	1250
Centre IV	1240	300	800	360	960

   1st July → 31st December

	B	F	FL	G	GG
Centre I	2500	1200	1280	950	2000
Centre II	1200	850	650	540	1240
Centre III	2200	950	1500	640	1450
Centre IV	950	640	720	480	820

   The company predicts that at each shop the sales for the next year will increase by 10% due to a new sales campaign. Assuming this prediction is indeed correct produce a 4 × 5 matrix showing the number of bags of each fertiliser sold at each shop for the following 1st January → 31st December. (Use a spreadsheet or the ability of some calculators to manipulate matrices if you wish.)

13. If a_{mn} is the element situated in the m^{th} row and n^{th} column of matrix A write down matrix A given that it is a 3 × 3 matrix with $a_{mn} = 2m + n$.

14. If a_{mn} is the element situated in the m^{th} row and n^{th} column of matrix A write down matrix A given that it is a 3 × 4 matrix with $a_{mn} = m^n$.

Multiplying matrices.

An inter-school sports carnival involves five schools competing in seven sports.

In each of these sports, medals, certificates and team points are awarded to teams finishing 1st, 2nd or 3rd.

The 5 × 3 matrix on the right shows the number of first, second and third places gained by each of the five schools.

	1st Place	2nd Place	3rd Place
School A	1	1	1
School B	3	1	0
School C	0	3	3
School D	1	2	0
School E	2	0	3

Suppose that points are awarded using the points system:

1st	3 points
2nd	2 points
3rd	1 point

The total points scored for each school are:

School A	School B	School C	School D	School E
1 × 3 +	3 × 3 +	0 × 3 +	1 × 3 +	2 × 3 +
1 × 2 +	1 × 2 +	3 × 2 +	2 × 2 +	0 × 2 +
1 × 1	0 × 1	3 × 1	0 × 1	3 × 1
6	11	9	7	9

Thus school B finished first with 11 points, followed by schools C and E equal second, school D was fourth and school A was fifth.

Note the way that each row of the 5 × 3 matrix is "stood up" to align with the points matrix. This is indeed how we carry out matrix multiplication.

Matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

If $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$ the product AB is found as shown below.

Follow each step carefully to make sure you understand where each element in the final answer comes from.

First spin the 1st row of A to align with 1st column of B, multiply and then add:

Thus $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{4} & -1 \\ \mathbf{1} & -3 \end{bmatrix} = \begin{bmatrix} (2)(1) + (1)(4) + (3)(1) \\ \end{bmatrix}$

Continue to use the first row of A, this time going *further across* to align with the 2nd column of B. We similarly go *further across* to place our answer:

$$\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{2} \\ 4 & \mathbf{-1} \\ 1 & \mathbf{-3} \end{bmatrix} = \begin{bmatrix} 9 & (2)(2) + (1)(-1) + (3)(-3) \\ \end{bmatrix}$$

Having "exhausted" the 1st row of A we now move *down* to use the 2nd row and similarly move *down* to place our answer:

$$\begin{bmatrix} 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 2 \\ \mathbf{4} & -1 \\ \mathbf{1} & -3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ (0)(1) + (-1)(4) + (2)(1) & \end{bmatrix}$$

Continuing the process:

$$\begin{bmatrix} 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{2} \\ 4 & \mathbf{-1} \\ 1 & \mathbf{-3} \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -2 & (0)(2) + (-1)(-1) + (2)(-3) \end{bmatrix}$$

Thus $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -2 & -5 \end{bmatrix}$

Confirm this result using your calculator.

Using your calculator to determine the product of matrices can be useful but if the matrices are not too big you should be able to determine the answers mentally. You would not need to show each step of the process and, with practice, you should be able to write the answer directly, as shown at the top of the next page.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -6 \\ -2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 13 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \end{bmatrix}$$

As was mentioned earlier, this method of matrix multiplication means that:

> Two matrices can be multiplied together provided the number of columns in the first matrix equals the number of rows in the second matrix.

Suppose matrix A has dimensions $m \times n$ and matrix B has dimensions $p \times q$.

- The product $A_{mn}B_{pq}$ can only be formed if $n = p$.
  In this case AB will have dimensions $m \times q$.
- The product $B_{pq}A_{mn}$ can only be formed if $q = m$.
  In this case BA will have dimensions $p \times n$.

Note: In the product AB we say that B is *premultiplied* by A
or that A is *postmutiplied* by B.

Example 2

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly and explain why.

(a) AB (b) BA (c) AC (d) CA (e) B^2

(a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ which cannot be determined because the number of columns in A ($2 \times \mathbf{3}$) ≠ the number of rows in B ($\mathbf{2} \times 2$).

(b) $BA = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 8 & 6 \\ 2 & 2 & 9 \end{bmatrix}$

(c) $AC = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \end{bmatrix}$

(d) $CA = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 0 \end{bmatrix}$ which cannot be determined because the number of columns in C ($3 \times \mathbf{1}$) ≠ the number of rows in A ($\mathbf{2} \times 3$).

(e) $B^2 = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix}$

Confirm the above answers using your calculator.

Example 3

A manufacturer makes three products A, B and C, each requiring a certain number of units of commodities P, Q, R, S and T. Matrix X below shows the number of units of each commodity required to make one of each product.

$$\begin{array}{l} \\ \text{Product A} \\ \text{Product B} \\ \text{Product C} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 3 \end{bmatrix} \end{array} = X$$

(a) Each unit of P, Q, R, S and T costs the manufacturer \$200, \$100, \$50, \$400 and \$300 respectively. Write this information as matrix Y which should be either a column matrix or a row matrix, whichever can form a product with X.

(b) Form the product referred to in (a) and explain what information it displays.

(a) As a column matrix, Y would have dimensions 5 × 1.

As a row matrix, Y would have dimensions 1 × 5.

Matrix $X_{3\times5}$ can form a product with $Y_{5\times1}$: $X_{3\times5}\,Y_{5\times1} = Z_{3\times1}$

$$\text{Thus } Y = \begin{bmatrix} \$200 \\ \$100 \\ \$50 \\ \$400 \\ \$300 \end{bmatrix} \begin{array}{l} \leftarrow \text{Cost of 1 unit of P} \\ \leftarrow \text{Cost of 1 unit of Q} \\ \leftarrow \text{Cost of 1 unit of R} \\ \leftarrow \text{Cost of 1 unit of S} \\ \leftarrow \text{Cost of 1 unit of T} \end{array}$$

(The order P, Q, R, S, T being consistent with the order in X.)

(b) $$XY = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \\ 50 \\ 400 \\ 300 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 1400 \\ 1150 \end{bmatrix} \begin{array}{l} \leftarrow \text{total commodity cost (\$) for producing 1 unit of product A} \\ \leftarrow \text{total commodity cost (\$) for producing 1 unit of product B} \\ \leftarrow \text{total commodity cost (\$) for producing 1 unit of product C} \end{array}$$

Note ☞ Whilst this chapter has considered

adding and subtracting matrices,

multiplying a matrix by a scalar

and multiplying matrices

the concept of dividing one matrix by another is undefined for matrices.

Exercise 6B

Determine each of the following products. If any are not possible state this clearly and explain why.

1. $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0{\cdot}5 & -0{\cdot}5 \\ -0{\cdot}5 & 1{\cdot}5 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 5 \\ 5 & 1 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ -3 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}$

19. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ determine the following:

(a) AB (b) BA (c) A^2 (d) B^2

20. Multiplication of numbers is commutative, i.e. if x and y represent numbers then xy is always equal to yx. Is matrix multiplication commutative for all pairs of matrices for which the necessary products can be formed? Justify your answer.

21. Provided the necessary products can be formed, matrix multiplication is associative, i.e. (AB)C = A(BC).

Verify this for (a) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$.

22. Provided the necessary sums and products can be formed, the distributive law:

$$A(B + C) = AB + AC$$

holds for matrices.

Verify this for (a) $A = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

23. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and k is a number, prove that

$$(kA)B = A(kB) = k(AB)$$

24. A is a 3 × 2 matrix, B is a 3 × 2 matrix, C is a 2 × 3 matrix and D is a 1 × 3 matrix. State the dimensions of each of the following products. For any that cannot be formed state this clearly.

(a) AB (b) BA (c) BC (d) CB
(e) AD (f) DA (g) BCA (h) DAC

25. With $A = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ state whether each of the following products can be formed or not.

(a) AB (b) BA (c) AC (d) CA
(e) BD (f) DB (g) AD (h) DA

26. If it is possible to form the matrix product AA (i.e. A^2) what can we say about A?

27. BC is just one product that can be formed using two matrices selected from the three below. List all the other products that could be formed in this way. (The selection of the two matrices can involve the same matrix being selected twice.)

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

28. (a) Premultiply $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

(b) Postmultiply $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ by $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

29. The 5 × 3 matrix shown on the right appeared on an earlier page. It shows the number of first, second and third places gained by each of five schools taking part in an inter-school sports carnival involving seven sports. Determine the rank order for these schools using the points

	1st Place	2nd Place	3rd Place
School A	1	1	1
School B	3	1	0
School C	0	3	3
School D	1	2	0
School E	2	0	3

matrix (a)

1st	5 points
2nd	3 points
3rd	1 point

(b)

1st	4 points
2nd	3 points
3rd	2 points

30. A financial adviser sets up share portfolios for three clients. Each portfolio involves shares in 4 companies with the number of shares as shown below.

	Able Co.	Big Co.	Con Co.	Down Co.
Client 1	1000	5000	400	270
Client 2	500	8000	500	250
Client 3	500	3000	500	500

Initially the value of each share is:

Abel Co.	$5
Big Co.	50 cents
Con Co.	$12
Down Co.	$10

Two years later the value of each share is:

Abel Co.	$4
Big Co.	60 cents
Con Co.	$20
Down Co.	$10

Use matrix multiplication to determine the value of each client's portfolio at each of these times.

31. A fast food outlet offers, amongst other things, Snack Packs and Family Packs. The contents of each of these are as in the contents matrix below:

	Drink (mLs)	Number of Burgers
Each Snack Pack	375	1
Each Family Pack	1250	4

An order comes in for 15 Snack Packs and 10 Family Feasts. Use matrix multiplication to determine a matrix that shows the total volume of drink and the total number of burgers this order requires.

32. Three hotels each have single rooms, double rooms and suites. The number of each of these in each hotel is as shown in matrix P below:

	Hotel A	Hotel B	Hotel C
Single	15	5	5
Double	25	25	14
Suite	2	1	3

= Matrix P

The three hotels are all owned by the same company and all operate the same pricing structure as shown in the tariff matrix, Q, shown below:

	Single	Double	Suite
Cost per night	\$75	\$125	\$180

= Matrix Q

(a) Only one of PQ and QP can be formed. Which one?

(b) Determine the matrix product from part (a) and explain what information it is that this matrix displays.

(c) Suppose instead that the tariff matrix were written as a column matrix, R. The matrix product PR could be formed but would it give any useful data? Explain your answer.

33. A carpenter runs a business making three different models of cubby house for children. Each cubby house is made using four different sizes of treated pine timber. The number of metres of each size of timber required for each cubby house is shown below.

	Poles 120 mm diameter	Decking 90 mm × 22 mm	Framing 70 mm × 35 mm	Sheeting 120 mm × 12 mm
Cubby A	3	30	20	40
Cubby B	4	35	25	60
Cubby C	6	40	30	70

We will call this matrix P.

(a) The carpenter receives an order for 3 type A's, 1 type B and 2 type C's. Write this information as matrix Q which should be either a row matrix or a column matrix, whichever can form a product with P.

(b) Determine the product referred to in part (a) and explain what this matrix represents.

(c) The poles cost \$4 per metre, the decking \$2 per metre, the framing \$3 per metre and the sheeting \$1·50 per metre. Write this information as matrix R which should be either a row matrix or a column matrix, dependent on which will form a product with P. What dimensions would this product matrix be and what information would it display?

34. A manufacturer makes four different models of a particular product. Matrix D below gives the number of units of commodities A, B and C required to make one of each model type.

	Model I	Model II	Model III	Model IV	
Commodity A	2	3	1	2	
Commodity B	20	30	50	40	= D
Commodity C	2	1	3	2	

(a) Each unit of the commodities A, B and C costs the manufacturer $800, $50 and $1000 respectively. Write this information as matrix E, either a column matrix or a row matrix, whichever can form a product with D.

(b) Form the product referred to in (a) and explain the information it displays.

35. A manufacturer makes three different models of a particular item. Matrix P below gives the number of minutes in the cutting area, the assembling area and the packing area required to make each model.

	Cutting	Assembling	Packing	
Each model A	30	20	10	
Each model B	20	30	10	= P
Each model C	40	40	10	

The manufacturer receives orders for 50 A's, 100 B's and 80 C's.

We could write this as a column matrix, Q: $\begin{bmatrix} 50 \\ 100 \\ 80 \end{bmatrix}$

or as a row matrix, R: $\begin{bmatrix} 50 & 100 & 80 \end{bmatrix}$

Both PQ and RP could be formed but only one of these will contain information likely to be useful.

(a) Which product is this?

(b) Form the product.

(c) Explain the information it gives.

Zero matrices.

Any matrix which has every one of its entries as zero is called a zero matrix.

Thus the 2 × 2 zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The letter O is used to indicate a zero matrix. If it is necessary to indicate that it is the zero matrix of a particular dimension, say 2 × 2 for example, then we write $O_{2 \times 2}$.

There are obvious parallels between zero in the number system and a zero matrix in matrices.

With x representing a number:

$$x + 0 = x, \qquad 0 + x = x, \qquad x \times 0 = 0, \qquad 0 \times x = 0.$$

With A representing a matrix, and providing the necessary sums and products can be formed:

$$\text{A} + \text{O} = \text{A}, \qquad \text{O} + \text{A} = \text{A}, \qquad \text{AO} = \text{O}, \qquad \text{OA} = \text{O}.$$

In this text we will use the letter O rather than the number 0 (zero) for the zero matrix. The two symbols can easily be confused, especially when handwritten. However this should not cause a problem, and you don't need to take time making some distinction between the handwritten characters, because the context usually makes it obvious whether it is the number zero or a zero matrix that is being referred to.

Because a zero matrix leaves another matrix "unchanged under addition", ie. A + O = A and also O + A = A, a zero matrix is sometimes referred to as an *additive identity matrix.*

Note: Care needs to be taken when working with matrices. We must guard against using rules and procedures that apply to numbers but that do not necessarily apply to matrices. For example, we have already seen that under matrix multiplication the matrix product AB is usually not the same as BA. Two more points to watch for are given below.

- With x and y representing numbers, a frequently used result in mathematics is that if $xy = 0$ then either $x = 0$ and/or $y = 0$.
  However, for matrices, if AB = O it is not necessarily the case that A and/or B = O.

  For example consider $\text{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$.

  In this case
  $$\text{AB} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  Thus AB = O but neither A nor B equal O.

- With x and y representing numbers, if $xy = zy$, for $y \neq 0$, then $x = z$.
  However, for matrices, if AB = CB, B ≠ O, matrix A is not necessarily equal to matrix C. i.e. We cannot simply cancel the B's.

  For example consider $\text{A} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$, $\text{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\text{C} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$.

  In this case
  $$\text{AB} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

  and
  $$\text{CB} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}.$$

  Thus AB = CB, B ≠ O, but A ≠ C.

Multiplicative identity matrices.

A multiplicative identity matrix leaves all other matrices unchanged under multiplication (provided the multiplication can be performed).

Thus if $I_{m \times n}$ is a multiplicative identity matrix then,

$$I_{m \times n} A_{n \times p} = A_{n \times p} \qquad \text{from which it follows that } m = n.$$

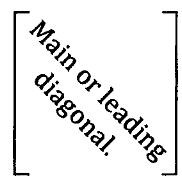
Also $$B_{q \times m} I_{m \times n} = B_{q \times m} \qquad \text{from which it follows that } m = n.$$

Thus multiplicative identity matrices are square matrices.

The letter I is used to indicate a multiplicative identity matrix. If clarification is needed as to the size of I we can write I_2 for the 2×2 multiplicative identity, I_3 for the 3×3 multiplicative identity etc.

If we simply refer to an identity matrix it should be assumed that it is a multiplicative identity matrix, I, that is being referred to.

A multiplicative identity matrix has every entry of its main or leading diagonal equal to one and every other entry equal to zero.



The 2×2 multiplicative identity matrix is: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The 3×3 multiplicative identity matrix is: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

There are obvious parallels between 1 in the number system and a multiplicative identity matrix in matrices.

With x representing a number: $1 \times x = x, \quad x \times 1 = x.$

With A representing a matrix: $I \times A = A, \quad A \times I = A.$

Again be careful not to use rules and procedures that apply to numbers but that do not necessarily apply to matrices.

In numbers, if $xy = x$, then for $x \neq 0$, y must equal 1.

However, for matrices, if $AB = B$, $B \neq 0$, A does not necessarily equal I.

For example $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$.

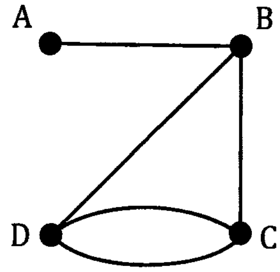
Premultiplication by $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$ has left $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ unchanged but $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} \neq I$.

Thus: Multiplication by I leaves a matrix unchanged, but a matrix being left unchanged under multiplication does not necessarily mean that it must have been I that we multiplied by.

I.e. For matrices A and B, even if we know that $AB = B$ we cannot assume that $A = I$, the identity matrix.

Route matrices

The diagram below shows the roads that exist between towns A, B, C and D.



Notice how the following statements about the road network are written in the *route matrix* shown.

From A there is 1 direct road to B and none to C or D.

From B there is a direct road to A to C and to D.

From C there is no direct road to A, 1 to B and 2 to D.

From D there is no direct road to A, 1 to B and 2 to C.

Note: By *direct* road from A to B we mean a road from A to B that does not pass through any of the other towns in between.

From \ To	A	B	C	D
A	0	1	0	0
B	1	0	1	1
C	0	1	0	2
D	0	1	2	0

The fact that all of the roads in the road system are "two way" means that:

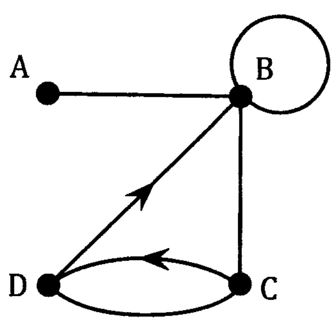
the number of roads from A to B is the same as from B to A,
the number from A to C is the same as the number from C to A,
the number from A to D is the same as the number from D to A, etc.

Can you see how this *symmetry* is also shown in the route matrix by the fact that
an entry in row *m* and column *n*
is the same as the entry in row *n* and column *m*.

Note also that because there are no direct roads from
A back to A (without passing through any of B, C or D in between),
B back to B,
C back to C
or D back to D
the leading diagonal of the route matrix contains only zeros.

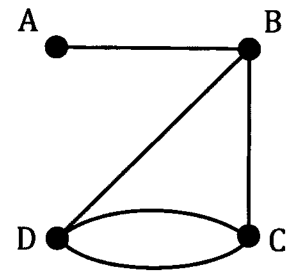
The network below left has some one way roads and a road that goes directly from B back to B. The route matrix now does not have the same symmetry as the previous one and the leading diagonal is now not all zeros.

Note: The matrix shows two direct routes from B to B because the road loop can be travelled clockwise or anticlockwise.

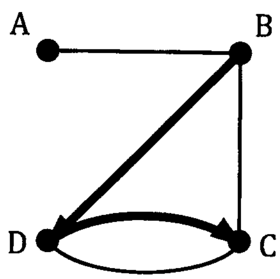


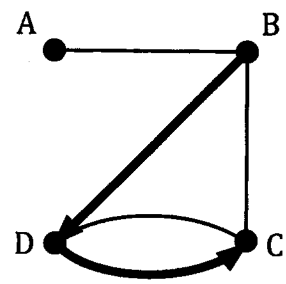
From \ To	A	B	C	D
A	0	1	0	0
B	1	2	1	0
C	0	1	0	2
D	0	1	1	0

Consider again the road system shown on the right.



Notice that there is only one way of travelling from A to A <u>in two steps</u>: A → B → A
and only one "two stage" route from A to C: A → B → C
but there are two "two stage" routes from B to C, as shown below:





Confirm that continuing this thinking leads to the *two stage route matrix* shown on the right.

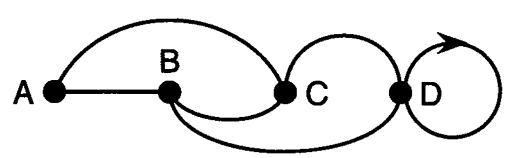
		To			
		A	B	C	D
From	A	1	0	1	1
	B	0	3	2	2
	C	1	2	5	1
	D	1	2	1	5

Either mentally or with the aid of your calculator calculate the following matrix product,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

i.e. (the one stage route matrix for this road system)^2, and compare your answer with the above two stage matrix.

Try to work out the two stage route matrix for the road network shown on the right and then check your answer by determining, and then squaring, the one stage route matrix.

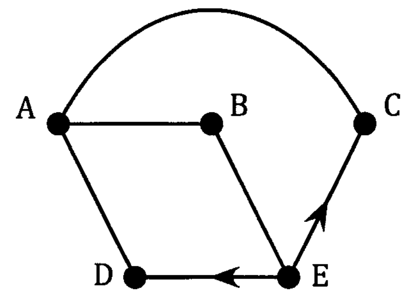


Note: We normally refer to a *one stage route matrix*, or *direct route matrix*, simply as a *route matrix*. If we mean the matrix showing the number of two stage routes or three stage routes we should call them *two stage route matrices* or *three stage route matrices*.

Social interaction as a matrix.

We can use a matrix to indicate the existence, or otherwise, of some form of social interaction between members of a group.

Suppose the diagram on the right indicates who, in a group of five people, has visited the house of somebody else in that group. The diagram shows, for example, that A has visited the house of B and of C and of D but has not visited the house of E,
E has visited the house of D (and of B and C) but D has not visited the house of B, C or E,
etc.



Using a "1" to indicate "has visited" and a "0" to indicate "has not visited" we can produce the matrix:

	A	B	C	D	E
A	0	1	1	1	0
B	1	0	0	0	1
C	1	0	0	0	0
D	1	0	0	0	0
E	0	1	1	1	0

Note:

☞ In this situation we can reasonably assume that everyone has visited their own house, although these "loops" are not included on the diagram. As this "has visited your own house" is really rather pointless information we simply place zeros on the leading diagonal. On some occasions like this we might simple decide to put dashes on the leading diagonal. (However if we later wanted to multiply the matrix by a scalar or another matrix "dashes" could cause a problem.)

☞ In the above matrix we have used the convention that the matrix shows "what the row person does to the column person". For example the entry in row 5 (person E) and column 4 (person D) shows that person E has been to the house of person D.

Exercise 6C

1. If $A = \begin{bmatrix} 2 & 0 \\ -4 & -3 \end{bmatrix}$ find (a) matrix B given that $A + B = O$, the 2×2 zero matrix.

   (b) matrix C given that $A + C = I$, the 2×2 identity matrix.

2. If $D = \begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix}$, O is the 2×2 zero matrix and I is the 2×2 identity matrix find

   (a) matrix E given that $DO = E$,
   (b) matrix F given that $D + O = F$,
   (c) matrix G given that $D + I = G$,
   (d) matrix H given that $DI = H$,
   (e) matrix J given that $ID = J$.

3. For each of the following, state whether the given statement is necessarily true for all matrices A, B and C for which the given operations can be determined.

(a) AI = A (b) IA = A (c) AB = BA

(d) 0A = 0 (e) A + B = B + A (f) A + A = 2A

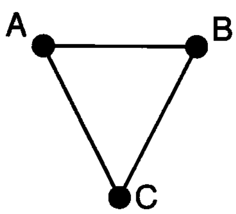
(g) A(B + C) = AB + AC (h) (AB)C = A(BC)

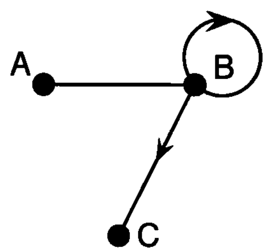
(i) If AB = 0 then A = 0 and/or B = 0.

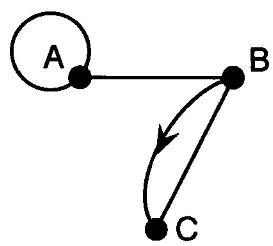
(j) If AB = AC and A ≠ 0 then B = C.

4. Determine the route matrix for each of the following road systems giving your answers in the form shown on the right.

To
A B C
From A B C [ ]

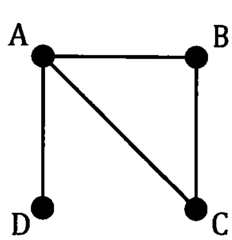
(a)


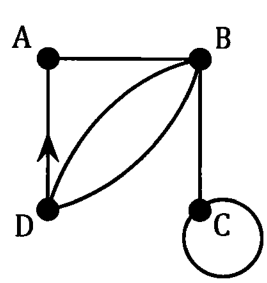
(b)


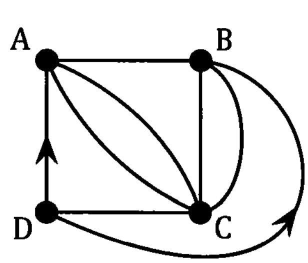
(c)


5. Determine the route matrix for each of the following road systems giving your answers in the form shown on the right.

To
A B C D
From A B C D [ ]

(a)


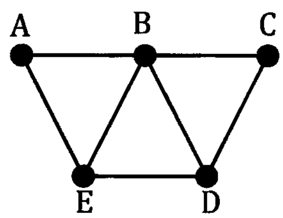
(b)


(c)


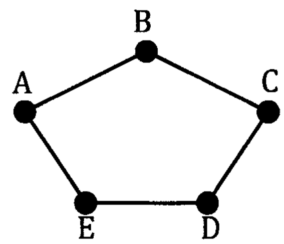
6. Determine the route matrix for each of the following road systems giving your answers in the form shown on the right.

		To				
		A	B	C	D	E
	A					
	B					
From	C					
	D					
	E					

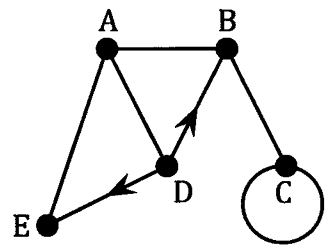
(a)



(b)

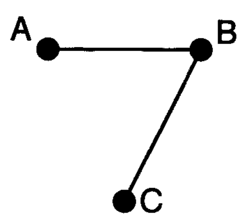


(c)

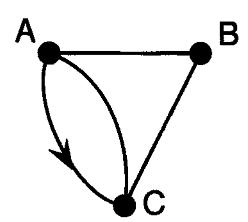


7. For each of the following write down the two stage route matrix and then check your answer by calculating, and then squaring, the direct route matrix.

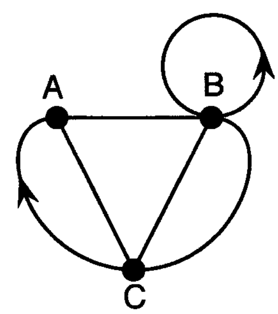
(a)



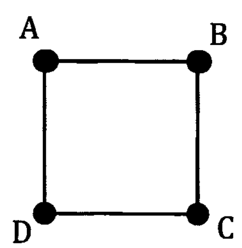
(b)



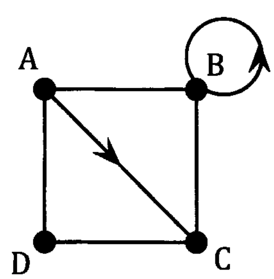
(c)



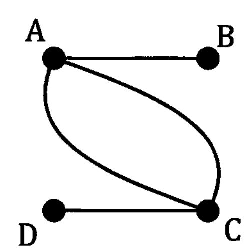
(d)



(e)

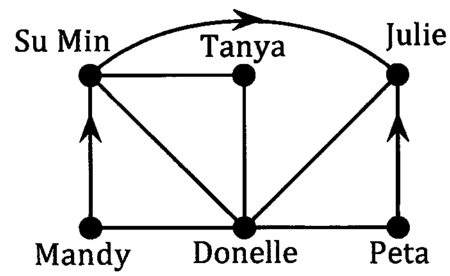


(f)



8. Investigate three stage route matrices.

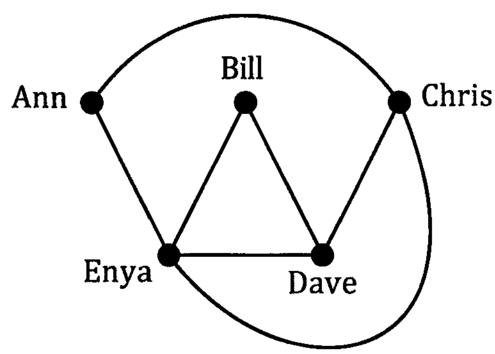
9. A group of six people took part in a survey in which they were asked, in confidence, to list who of the other five people in the group they considered "close friends". An analysis of the responses gave the diagram on the right.
This diagram shows, for example that Donelle considers Julie a close friend and this feeling is reciprocated with Julie considering Donelle a close friend.
However, whilst Mandy considers Su Min a close friend this feeling is not reciprocated by Su Min , who does not consider Mandy a close friend.



These examples are also shown in the matrix on the right.
Placing zeros on the leading diagonal to make the "being a close friend of oneself" count zero, copy and complete the matrix.

	Sue Min	Tanya	Julie	Peta	Donelle	Mandy
Sue Min						0
Tanya						
Julie					1	
Peta						
Donelle			1			
Mandy	1					

10. The diagram on the right shows "who has been to the movies with who in a particular month" amongst a group of five friends, Ann, Bill, Chris, Dave and Enya.

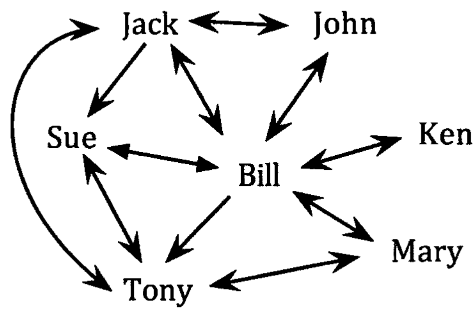


(a) Why are there no "one way paths" in the diagram?
(b) What effect will this "no one way paths" situation have on the matrix showing this "been to the movies with each other" relationship?
(c) Construct the matrix of this situation using
- zeros to show a pairing has not been to the movies together in the month,
- ones to show that a pairing has been to the movies together in the month,
- zeros on the leading diagonal so that the situation of accompanying yourself to the movies is not counted.

(d) During a second month this group of five people were involved in three movie trips. On one of these Ann, Bill and Enya went together,
on another Bill and Chris went together,
and on another Dave went with Enya.
Construct a matrix like the one you did for part (c) for this second month.

11. Explain why it could be useful to be able to display social interactions using numbers in a matrix.

12. The diagram on the right shows "who has who's phone number" in a group of seven school students.
The arrows indicating, for example, that Jack has John's and John has Jack's but whilst Jack has Sue's, Sue does not have Jack's.
The matrix for this social interaction is started below with



- the usual convention observed in that the row person "does the activity" to the column person. i.e. the cell a_{mn} shows whether or not m has the phone number of n,
- zeros used to indicate not having the number and ones used to indicate having the number,
- zeros placed in the leading diagonal so that the possession of one's own phone number will not count.

	Jack	John	Sue	Bill	Ken	Tony	Mary
Jack	0	1	1	1	0	1	0
John		0					
Sue			0				
Bill				0			
Ken					0		
Tony						0	
Mary							0

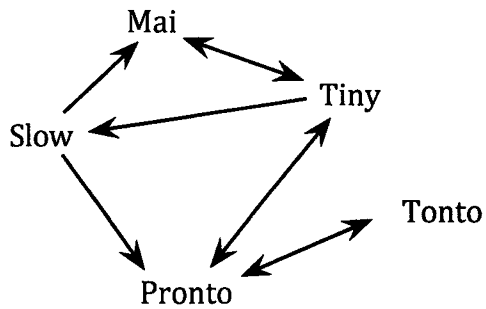
(a) Copy and complete the above matrix.

(b) Although Jack does not have Mary's number he could try to get it using one of the two-stage links Jack → Bill → Mary
Jack → Tony → Mary.
Copy and complete the following two stage matrix and then see how your answer compares with the matrix obtained by squaring your part (a) answer.

	Jack	John	Sue	Bill	Ken	Tony	Mary
Jack	0						2
John		0					
Sue			0				
Bill				0			
Ken					0		
Tony						0	
Mary							0

(c) Are there any of the seven who do not have the number of one of the others AND cannot obtain that number by a two stage process?
What feature of your two matrices shows this?

13. As part of the security system the top five agents in an undercover police operation can only initiate direct contact with some of the other four, but not all.
For example, as the diagram below shows, "Slow" can initiate direct contact with "Mai" and "Pronto" but can only have direct contact with her initiated by "Tiny"



(a) Using the order as in the matrix on the right determine the one stage and two stage matrices for this situation with:

	Mai	Tiny	Tonto	Pronto	Slow
Mai					
Tiny					
Tonto					
Pronto					
Slow					

- the usual convention observed in that the row person "does the activity" to the column person,
- placing one of just two possible entries in each space, either a zero (0) to indicate no contact or a one (1) to indicate contact,
- zeros placed in the leading diagonal on both the one stage and two stage matrices so that being able to contact oneself, either as a one step or two step process is not counted.

(b) Is it the case that your two stage matrix is the square of your one stage matrix? If not, explain the differences.

Miscellaneous Exercise Six.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary section at the beginning of the book.

1. Matrix W_{mn} has m rows and n columns. For each of the following state whether the calculation can be performed. If it cannot be, then state this clearly, and if it can be, state the number of rows and columns the final matrix would have.

(a) $A_{23} + B_{32}$	(b) $2 \times A_{23}$	(c) $B_{32} - C_{25}$	(d) $B_{32} \times C_{25}$
(e) $D_{31} \times E_{31}$	(f) $E_{31} \times F_{14}$	(g) $(F_{14})^2$	(h) $(G_{33})^2$
(i) $H_{21} \times J_{21}$	(j) $J_{21} - K_{12}$	(k) $L_{12} \times M_{21}$	(l) $(N_{54} + P_{54}) \times R_{43}$

2. If $A = \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$ determine each of the following:

(a) A + B (b) 2A (c) A – B

(d) AB (e) $(A - B)^2$ (f) $A^2 - B^2$

3. If a company does not have enough work to employ someone full time but would instead like to be able to call a worker in on the days they need them, the company may employ a worker as a *casual worker.* Casual work is usually temporary with no guarantee of it continuing long term and may involve irregular hours. A student working in a temporary position during college holidays would be a common example of casual work.

A company pays their permanent factory labourers $26·40 per hour. At certain times they need extra labourers and employ casual workers paying them 110% of the permanent rate, the higher rate reflecting the fact that casual workers have no paid sick leave, no long service leave and accumulate no holiday pay.

Calculate the weekly pay for each of the following people if the hours shown are for one week, permanent work force are paid time and a half for hours worked over 35 hours per week and casual workers receive the basic casual rate for all hours worked.

(a) Jaun, permanent, 37 hours (b) Jackie, casual, 38 hours.

(c) Su-Lin, permanent, 45 hours (d) Ravinder, casual, 42 hours.

4. If an employee works *shiftwork* they can be *rostered* on to work their normal hours, eg 8 hours a day for 5 days a week, on any days from Monday to Sunday. This rostering could include some night. Nursing, for example, would be one area where shiftwork would be commonly used.

Suppose a shift worker is paid $32·80 per hour day rate (DR). Night and weekend hourly rate (NWR) is 130% of the day rate and public holiday loading is 150% of either the DR or NWR as appropriate.

Calculate the weekly pay for each of the following:

(a) Sheila, 19 hrs day duty + 8 hrs night duty + 8 hrs public holiday night duty.

(b) Tobias, 16 hrs day duty + 16 hrs night duty + 8 hrs public holiday day duty.

5. Copy and complete the following table to compare various forms of interest for a loan of $25 000 at 9% per annum, using technology as you deem appropriate.

	$25 000 borrowed at 9% per annum			
	Simple Interest	Compounded Annually	Compounded every 6 months	Compounded quarterly
Initial amount borrowed				
Amount owed after 1 year				
Amount owed after 2 years				
Amount owed after 3 years				
Amount owed after 4 years				
Amount owed after 10 years				
Amount owed after 20 years				